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Temporal development of invasion percolation

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Abstract. We consider the dynamics of invasion percolation for different models. This process exhibits scaling behaviours which, in the simplest case, can be related to some critical static properties of usual percolation. In particular, the concept of bursts is shown to be of fundamental importance. The critical exponents we derive are checked numerically on a case of invasion percolation where growth is restricted to the hull of the existing cluster.

1. Introduction

Percolation theory [1] is one of the most significant breakthroughs in the advance of statistical physics into the field of disordered material science. However, *a priori*, very few materials are likely to be close to the percolation critical point, since it corresponds to a very narrow region of the control parameter (concentration of one phase in a composite material, proportion of links between particles, time or temperature in a sol-gel reaction, etc).

However, there are at least two examples where percolation can be applied naturally, without making use of very restrictive hypotheses on the disorder. The first case is that of an exponentially wide distribution of local transport properties where a percolation-like sublattice is known to control the global properties [2]. The second example, which will concern us in this paper, is a member of a large class of phenomena recently highlighted [3] under the generic name of 'self-organised criticality'. Indeed, the construction of a percolation infinite cluster can be given a dynamic such that the percolation point appears to be a *stable fixed point*. Therefore, in such a case, the system will naturally evolve toward its critical percolation point. *Invasion percolation* [4] falls into this category.

Apart from this general property, invasion percolation is relevant for immiscible fluid displacement in porous media [5]. A wetting fluid initially saturating a porous medium is progressively replaced by a non-wetting one injected from one side of the medium, under pressure, at a very low flow rate (where capillary effects are dominant compared with viscous ones). Due to the one-to-one correspondence between the capillary pressure at the limit of separation of the phases in the pores and the random distribution of pore radii, the injection process generates a growing percolation cluster when the amount of non-wetting fluid increases with pressure (a control parameter). In particular, the breakthrough corresponds to the pressure value when the invading fluid extends at very large distances from the injection wall. It has been recently

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emphasised experimentally [6] and numerically [7]† that the invasion takes place with irregular jumps associated with the filling of pockets connected to the invaded cluster by a narrow neck. This is the basis of 'Haines jumps' [9] or 'bursts', whose role will be underlined in the understanding of the temporal development of invasion percolation. In practice, it is possible to stabilise the percolation cluster to its critical point as discussed above provided the capillary medium does not have a capillary barrier at its exit side: in this case, the invading fluid will start flowing freely out of the medium as soon as it reaches this border without further invasion.

A recent numerical study by Furuberg *et al* [10] has revealed an important scaling feature of the dynamics of invasion percolation which emphasises the key concept of *bursts*, as we will see below. More precisely, the distribution, $N(r, T)\,dr$, of distances, r, between two sites added to the invaded cluster and separated by a time interval of T (i.e. one site is picked at a spatial coordinate x and a time t, while the other is picked at $x+r$, and at a time $t+T$) was found to follow the scaling law

$$N(r, T) = r^{-1}\varphi(r^D/T) \tag{1}$$

where the function φ has the two limiting behaviours:

$$\varphi(x) \propto x^a \qquad \text{for } x \ll 1 \tag{2}$$

$$\varphi(x) \propto x^{-b} \qquad \text{for } x \gg 1 \tag{2'}$$

with $a = 1.4$ and $b = 0.6$. D refers to the fractal dimension of the invaded cluster, measured to be about 1.82 in this case [10].

We propose, in this paper, to connect the exponents a and b appearing in this scaling form with known exponents of usual percolation. We will consider here invasion percolation as a specific procedure to generate a cluster, with rules to be defined precisely below, without addressing the question of relevance to a real experimental phenomenon. We will nevertheless use freely the terms arising from such a correspondence: pressure, volume, incompressibility, etc.

2. Models

Let us consider a lattice whose sites, i, are assigned random independent numbers, x_i, uniformly sampled on [0, 1]. We start with one side of the lattice as the 'invaded cluster'. Then we look among the neighbouring sites of this cluster (the 'growth sites') for the one which carries the smallest random number, x, (the 'pressure'). This site is then added to the cluster, and the new sites susceptible to growth are changed accordingly. The number of sites of the invaded cluster—its volume or mass—is very often considered as a 'time' parameter. Physically, it would correspond to an invasion performed at constant flow rate. This is the justification for the term 'dynamics' to qualify the development of the structure. The above rule defines the simplest invasion percolation, which we will refer to as model I.

It is also important to modify the previous algorithm to take into account the incompressibility of the displaced fluid [4]. In this case, at each step of the process,

† More precisely, these numerical studies were concerned with the occurrence of power law distributions in the conductance and resistance jumps during the invasion process. The origin of these distribution can be tracked back to the existance of a multifractal spectrum of the studied quantities *at* the percolation threshold [8]. However, the occurrence of bursts can affect the experimental measurement of the distributions and thus eventually account for the discrepancy of results obtained in [6, 7].

the sites that are no longer accessible from the opposite border are removed from the set of growth sites since the invaded fluid is trapped into such sites. However, this second version is not uniquely defined unless we specify further details. Two connectivity criteria must be used.

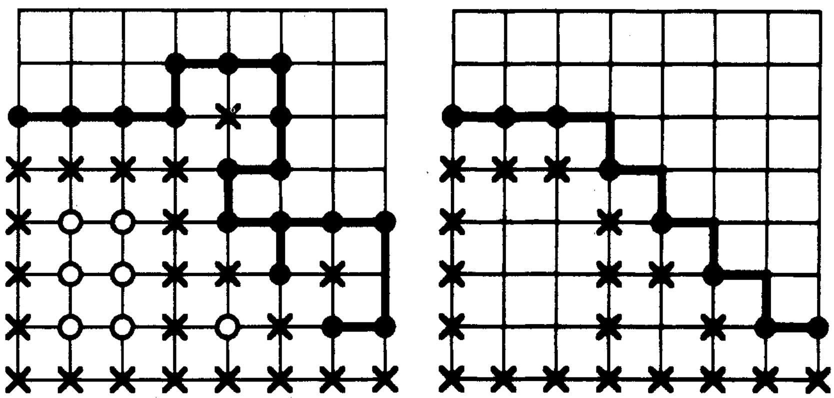
The first criterion is for the occupied sites of the 'invaded cluster'. We deliberately used here the word 'neighbouring in a vague sense. We can choose this neighbourhood to be restricted to nearest neighbours, or we can enlarge it to second neighbours too.

The second criterion is for the empty sites. A site A must be accessible from the opposite border in order for the invaded fluid to escape freely. Thus, there should exist a continuous path of empty sites which connects A to this border. The continuity implies a definition of what we consider to be connected.

The choice of the continuity rules for both phases at their common boundary depends in practice on the local hydrodynamics and on the wetting properties as was analysed experimentally with great care by Lenormand *et al* [11]. We now consider this choice at the theoretical level.

The most natural choice is to use for the set of empty sites a continuity definition which is *complementary* to that of the set of occupied sites. (A complementary set of connectivity definitions is such that if there exists a continuous path according to one connectivity rule which spans the system in one direction, there does not exist any continuous path transversely, and if such a latter path exists then the former does not.) For instance, on a square lattice, if one definition of connectivity is between nearest neighbours, then the complementary rule will include not only nearest neighbours but also diagnonal next-nearest neighbours. In this case, the set of growth sites will be the 'hull' of the invaded cluster (see figure 1(a)). This will be referred to as model II.

However, it is also possible to have definitions of continuity for the two phases which are not complementary (for instance, continuity is between nearest neighbours for both classes of sites on a square lattice). In this case, the set of growth sites will



(*a*) (*b*)

Figure 1. (a) The connectivity of invaded sites (×) of this square lattice is *via* nearest and diagonal second neighbours. The different growth sites are shown for the first two models considered in the text: model I accepts all frontier sites (● and ○) on a lattice with only nearest-neighbour connections (interior and exterior); model II is limited to the 'hull' (bold line) of the cluster and excludes the filling of holes (○) in order to fulfil the incompressibility requirement. (b) The connectivity of invaded sites (×) is limited to nearest neighbours. The growth sites (●) of model III are restricted to the external perimeter.

be the intersection of the 'unoccupied external perimeter' of the cluster of invading fluid and of the 'unoccupied internal perimeter' of the cluster of displacing fluid containing the opposite border (see figure 1(*b*)). This case will be referred to as model III.

The distinction between these last two models might seem formal, but it is not! A simple way to stress the importance of this difference, is to note that the fractal dimension of the hull of a percolation cluster D_h is 7/4 [12] in two dimensions, whereas that of the external perimeter, D_e, is 4/3 [13]. Therefore, the third model is much more restrictive than the second one on the set of available growth sites.

3. Connection with a percolation problem

It is natural to compare the invasion percolation cluster with a percolation cluster obtained by considering as occupied all sites carrying a random number smaller than a given value, p, and connected to the initial border. With this reference, the plain invasion percolation (model I) matches from time to time exactly the conformation of the usual incipient infinite percolation cluster (the isolated finite clusters are not considered). More precisely, this will happen each time the random number, $x(t)$, picked on the site that is invaded at time t, is larger than all previous $x(t')$ for $t' < t$. Thus, we expect the properties of this invasion percolation to be simply related to those of usual percolation. For instance, the fractal dimension of the invaded cluster should be that of the infinite cluster in percolation or, in two dimensions,

$$D_f = 91/48 \simeq 1.89. \tag{3}$$

For the other two models, such a correspondance is no longer valid. However, we note a strict inclusion which implies that the fractal dimension of the invaded cluster, D, decreases when we go from model I to II, and to III. The numerical results of Wilkinson and Willemsen [4] give the estimate $D \simeq 1.82$ for model III, which indeed suggests a strict inequality. This result is confirmed by the numerical work of Furuberg *et al* [10]. We will see below that model II seems close to the first case ($D \simeq 1.86$).

We will establish the connection with usual percolation, first using model I. We will then discuss the differences with other models.

In an invasion experiment, the measurable quantities are the pressure, the injected volume (or here equivalently the time). Therefore, a meaningful correspondance with a percolation, p (concentration of occupied sites), with an observable quantity, say the time. We can introduce the usual control parameter, p, via the correlation length, ξ,

$$\xi \propto (p - p_c)^{-\nu} \tag{4}$$

where p_c is the percolation threshold, and the scaling of the mass (time) of the invaded cluster. For a lattice of size L, the mass, t, of the percolation cluster can be written:

$$t \propto (L/\xi)\xi^D. \tag{5}$$

We obtain the 'effective' distance to threshold as

$$(p - p_c) \propto t^{-e} \tag{6}$$

where

$$e = 1/\nu(D-1). \tag{7}$$

An indirect way of checking this relation is to compute the equivalent equation of (4) for the number of growth sites, N_g, using the fractal dimension of this set at the breakthrough point

$$N_g \propto (L/\xi)\xi^{D_g} \tag{8}$$

or

$$N_g \propto t^g L^{(1-g)} \tag{9}$$

where

$$g = \frac{(D_g - 1)}{(D-1)}. \tag{10}$$

At a given time t, the distribution of random numbers x of sites already incorporated to the cluster is uniform on $[0,p]$, except very close to the upper limit, p, of the interval. However, the complete sequence $x(t)$ contains much correlation between consecutive numbers. This can also be seen from the fact that the growth process appears through bursts of sites close to each other. It is the structure of these bursts that was the origin of the scaling observed in the dynamics.

These bursts come from the accessibility of new sites once the set of growth sites has evolved. A burst can be defined in two ways. The first definition consists in calling a burst a connected cluster of sites added during a continuous time period. More precisely, if the set of sites invaded during the time interval $[t, t+T]$, form a connected cluster then it is a burst. This definition is certainly the most natural; however, it is difficult to use experimentally (for instance, it requires a two-dimensional transparent sample) and numerically (it requires a large amount of computation time). This definition has, however, been used in some cases [14]†.

We will not use that definition in the following but, instead, a second one, which forgets the notion of connectivity and focuses only on the signal $x(t)$. Bursts appear there as 'valleys'. One interesting feature of the series $x(t)$ is that it explicitly gives the hierarchical structures of large valleys inside which are carved some smaller ones, which themselves contain still smaller ones, and so on. In order to take account of this aspect, we propose to define a burst corresponding to a time t as the time lapse θ such that:

$$x(t') < x(t) \qquad \text{for all } t' \text{ such that } t-\theta < t' < t \tag{11a}$$

and

$$x(t-\theta) > x(t) \tag{11b}$$

where $x(t)$ is the 'root' of this burst. The construction is illustrated in figure 2. These bursts can be termed 'backward' since we look backward in time from the root. The analogous 'forward bursts' have the same statistics. Physically, once the site i_0 has been included into the cluster at time t_0, it allows growth to proceed to its neighbouring sites. If the subsequent sequence of $x(t)$ up to a time t_1 is lower than $x(t_0)$, then the corresponding sites form a cluster necessarily connected to the root i_0, otherwise growth would have occurred before t_0. Therefore, the intuitive notion of burst matches the one we use here. In addition, we note that our definition can be easily applied to an

† The notion of bursts and the scaling of N_τ as well as some numerical simulations are given in [14] by Nadal. The definition of burst used in this paper is slighly different.

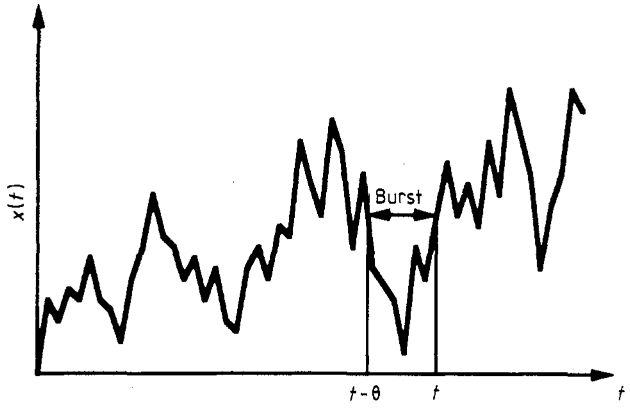


Figure 2. The concept of burst can be easily defined through the diagram showing the random numbers picked on the invaded sites, $x(t)$, as a function of time. Starting at a time t, the burst size θ, is defined as the maximum time lapse, for which all $x(t')$ are smaller than $x(t)$, for $t' \in [t-\theta, t-1]$. The relation with a more intuitive geometrical definition of a burst is explained in the text.

experiment, since it does not require any geometrical information from the medium, but only makes use of the fluctuation of the pressure signal recorded during a constant flow rate injection. Moreover, the latter definition of 'bursts' is equivalent to the former in most cases (the rare cases where these two definitions disagree should not affect the scaling relations proposed in the following).

We now investigate the statistical distribution of these bursts. One particular burst, which starts from a root $x(t)$, can be identified with a finite percolation cluster which includes the root site, and where the control parameter p assumes the value $x(t)$. If we consider only the bursts whose root $x(t)$ is comprised between p and $p+\mathrm{d}p$, we can write directly the distribution of the burst $n(\theta, p)$, from the usual cluster-size distribution in percolation. We recall that [1] the number of clusters of size s per site, n_s, varies as

$$n_s = s^{-\tau} f(s^{\sigma}(p-p_c)) \tag{12}$$

Each cluster is counted only once in n_s, regardless to its size. One site being chosen, the probability that its cluster is of size s is proportional to n_s times the number, s, of sites of the cluster. In our case, we only consider those sites which are likely to be a root for the cluster. Since they are located on the hull of the cluster, their number is proportional to $s^{D_h/D}$. The distribution $n(\theta, p)$ can thus be written

$$n(\theta, p) = \theta^{D_h/D-\tau} f(\theta^{\sigma}(p-p_c)). \tag{13}$$

The complete burst distribution, $n(\theta)$, is the integral of $n(\theta, p)$ over p, with a weight such that we recover the distribution of x. Earlier, we have argued that x is uniformly spread between 0 and p_c. Thus

$$n(\theta) = \int_0^{p_c} \theta^{D_h/D-\tau} f(\theta^{\sigma}(p-p_c)) \frac{\mathrm{d}p}{p_c} \tag{14}$$

or

$$n(\theta) \alpha \theta^{-\tau'}. \tag{15}$$

We have a power law distribution with an exponent $\tau' = \tau + \sigma - D_h/D$†.

4. Dynamic scaling

We now turn to the scaling of the distribution of distances between growth sites separated by a time interval T. In order to derive the scaling form recalled in equation (1) [10], we first introduce two exponents τ'' and α. Their values will be considered in the following.

Given a time interval T, we first consider the burst size θ which is the smallest burst which encompasses the two sites rached at times t and $t - T$. We postulate that these burst sizes are distributed according to a power law:

$$P_T(\theta) \propto \theta^{-\tau''} \tag{16}$$

with $\theta > T$. Normalisation of this distribution provides the T dependence of the prefactor.

$$P_T(\theta) \propto (1/T)(\theta/T)^{-\tau''} \mathfrak{F}(\theta - T) \tag{16'}$$

where $\mathfrak{F}(x)$ is 1 if x is positive and 0 if not.

Once we know that the growth occurred in a burst of size θ, we consider the distribution, $Q_\theta(r)$, of distances, r, between growth sites in this burst size. We postulate here a second power law distribution valid for r smaller than $\theta^{1/D}$. Introducing the second exponent α, and taking into account the normalisation, we are led to

$$Q_\theta(r) \propto r^\alpha \theta^{-(1+\alpha)/D} \mathfrak{F}(\theta^{1/D} - r). \tag{17}$$

Let us first show that the two previous relations naturally lead to the scaling law written in (1). The statistical distribution of the distance distribution, $N(r, T)$, is given by the integral

$$N(r, T) = \int_0^\infty P_T(\theta) Q_\theta(r) \, d\theta. \tag{18}$$

Two situations can occur:

$$N(r, T) \propto \int_T^\infty r^\alpha T^{\tau''-1} \theta^{-(1+\alpha)/D} \theta^{-\tau''} \, d\theta$$

$$\propto r^\alpha T^{-(1+\alpha)/D} \qquad \text{if } r < T^{1/D} \tag{19}$$

$$N(r, T) \propto \int_{r^D}^\infty r^\alpha T^{\tau''-1} \theta^{-(1+\alpha)/D} \theta^{-\tau''} \, d\theta$$

$$\propto r^{(1-\tau'')D-1} T^{\tau''-1} \qquad \text{if } r > T^{1/D}. \tag{20}$$

These two equations can be written in the form of the scaling function of equation (1) with the identification:

$$a = (1 + \alpha)/D \tag{21}$$

$$b = \tau'' - 1. \tag{21'}$$

Equations (21) and (21') relate the scaling properties of the dynamics of invasion percolation to the structure of the bursts which characterise the growth. We are now left with the problem of finding the values of α and τ''.

† This result is consistent with a bound $\tau' > \tau + \sigma - 1$ first derived by Nadal [14].

5. Identification of the exponent τ''

The exponent τ'' (16) is related to the burst-size distribution which encompasses two growth sites separated by a time interval T. At each time t, two possible cases can happen: if the burst whose root is $x(t)$ is larger than T, then we keep it; otherwise, we look for the next burst which 'shells' the previous one, until we finally include $x(t-T)$. *Assuming* a decorrelation of the consecutive burst sizes in this second case, apart from the trivial inequality of sizes resulting from the hierarchical inclusion, we obtain that τ'' is equal to τ' (15), or

$$b = \tau + \sigma - D_h/D - 1. \tag{22}$$

In order to render plausible the assumption of decorrelation needed to obtain this result, we can relate this property to the cluster-size distribution close to the percolation threshold. As seen from (12), this distribution is a power law in s up to a cutoff which varies with the distance to threshold, $|p-p_c|$. When p is changed, say when $|p-p_c|$ is reduced, a fraction of the finite clusters will merge in such a way that only the cutoff is modified, but not the power law of the distribution itself. This is somewhat similar to the construction we use here. The hypothesis of decorrelation we use justifies the fact that the lower cutoff is an external parameter, T, but the power law is not affected by this limit.

Using the value of the exponents for usual percolation, we get exactly in two dimensions, $b=1/D$ or numerically, $b=48/91 \simeq 0.527$. Therefore, at short times, we expect a distance dependence of $N(r, T)$ as $(r^{-2}T^{1/D})$ from (20). The fact that the exponent of the distance r is exactly 2 certainly evokes the possibility of a simpler argument than the one we used to derive this result.

6. Identification of the exponent α

The exponent α is related to the distribution of distances between a pair of sites such that both are contained within one burst of size θ, and not any smaller one. A first naive guess would be to consider with an equal probability all sites of the cluster. Therefore, due to the fractal character of the cluster, one would then conclude that the probability to jump at a distance between r and $r+\mathrm{d}r$, is proportional to r^{D-1}. However, we recall that the burst size θ is the *minimal* burst which includes both sites. Two sites in close neighbourhoods are likely to belong to smaller bursts, thus compelling the two sites to be further away than with a uniform distribution. This acts as a repulsive term. We thus obtain $\alpha \geqslant D-1$ or $a \geqslant 1$ (using (21')). In fact, this repulsive effect should not affect the scaling for large distances (the 'repulsion' effect is negligible when r is of the order of the maximum distance available, or close to the crossover point), but it should certainly give strong correction to the power law behaviour at small distances. More quantitatively, the probability r^{D-1} should be weighted by the probability that the pair of sites are not contained with a smaller burst which should thus have the form $(1-Ar^u)$ (where A is a constant and u is a characteristic exponent). If this is true, then we would have

$$\alpha = D-1 \tag{23}$$

and

$$a = 1 \tag{23'}$$

at least close to the crossover point ($r \propto T^{1/D}$), with some downward curvature for smaller distances, at a given time interval T. Indeed, from the curves presented in figures 2 or 3 of the work of Furuberg *et al* [10], a clear curvature is visible, and an exponent a equal to 1 close to the apex is not ruled out, although their estimate of a over the whole range of values was larger ($a = 1.4$). We emphasise that this last estimate is rather speculative.

7. Numerical results

We have studied invasion percolation numerically using model II. This was done on a square lattice (see figure 1(a)). At each time step, the growth sites are identified by following the hull of the invaded cluster by a classic 'maze-exploration' algorithm. The lattice size considered varied from 20 to 80. Periodic boundary conditions were implemented in the direction parallel to the initial border line. The connectivity criterion used for the hull was restricted to the first neighbours. Therefore, the matching connectivity criterion for the invaded cluster was to the nearest and the diagonal next-nearest neighbour. The corresponding percolation threshold to be considered is thus $p_c = 1 - p^*$ where p^* is the usual site percolation threshold on the square lattice or, $p_c = 0.407\,254$. We indeed observed (figure 3) that the mean value of x saturates for long times around $p_c/2 \simeq 0.20$, consistent with a uniform distribution of x between 0 and p_c.

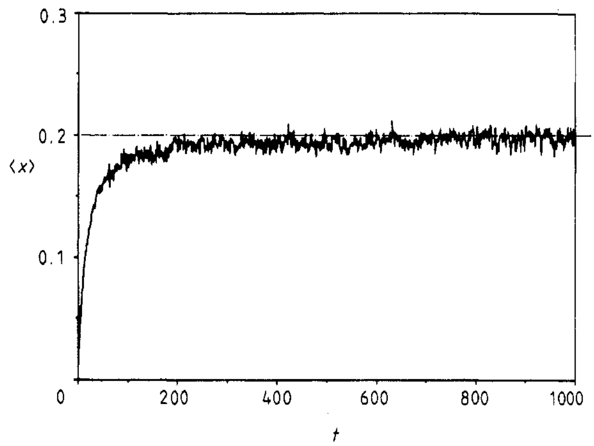


Figure 3. Evolution of the mean value of the parameter x as a function of time for lattices of size 80. This value saturates at a value $p_c/2$, consistent with a uniform distribution of x between 0 and p_c.

Figure 4 shows the mean mass of the invaded cluster when it reaches the opposite border, as a function of the lattice size. The slope of the best-fit line in a log–log plot gives the estimate of the fractal dimension:

$$D = 1.86 \pm 0.04 \tag{24}$$

quite consistent with the fractal dimension of a percolation cluster (3). The number of growth sites also varies as a power law (see figure 4). Our estimate is

$$D_g = 1.76 \pm 0.04 \tag{25}$$

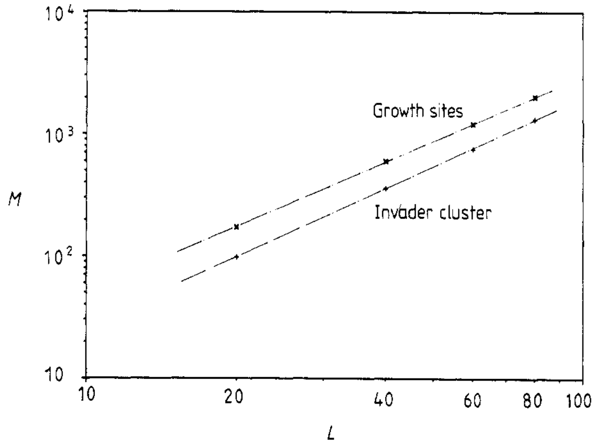


Figure 4. Log-log plot of the number of growth sites (upper curve), and of the mass (lower curve) of the invaded cluster (model II) when it reaches the opposite border of a square lattice, as a function of the lattice size, L. Each point is an average of a few hundred realisations. The best fits, indicated by the straight lines have a slope of 1.76 and 1.86 respectively, which should be compared with the corresponding values for usual percolation, $7/4=1.75$ and $91/48\simeq 1.89$.

very close to the value of the fractal dimension of the hull ($D_h=7/4$). We note that it is possible that the fractal dimension of the invaded cluster (for models II and III) is different from that of the infinite cluster in percolation, but the fractal dimension of the hull and of the external perimeter must be equal to their values in percolation, since the restriction on growth sites only affects the time counting, and the *internal* structure of the invaded cluster ('internal' refers to sites that are not on the external perimeter).

Figure 5 shows, in a log-log plot, the evolution of the number of growth sites as a function of time for a lattice size of 80. From (9), we expect a slope $g=36/43\simeq 0.837$,

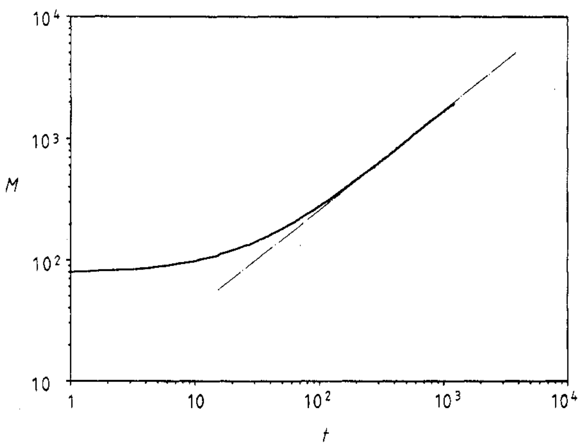


Figure 5. Log-log plot of the number of growth sites as a function of time for invasion percolation (model II), for lattices of size 80. The asymptotic behaviour, indicated by the straight line, has a slope of 0.82, which should be compared with the estimate 0.84 given by equations (9) and (10).

very close to the measured value

$$g = 0.82 \pm 0.04. \tag{26}$$

We now turn to the burst-size distribution. From the signal $x(t)$, we have extracted the backward burst-size distribution, as shown in figure 2. Figure 6 shows a log-log plot of this distribution. The measured slope is

$$\tau' = 1.50 \pm 0.04 \tag{27}$$

to be compared with our expected value: $\tau' = \tau + \sigma - D_h/D = 139/91 \simeq 1.527$.

We have also investigated the distance distribution for a unit time interval. The data are markedly affected by the small size of our lattice, and an abrupt drop for a distance roughly equal to the lattice size is clearly visible on the distribution shown in figure 7. However, the short distance behaviour is consistent with our prediction.

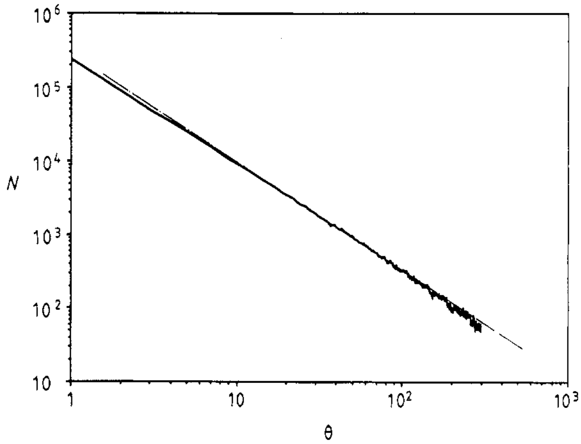


Figure 6. Log–log plot of the burst-size distribution, as illustrated in figure 2, for lattices of size 80. A best-fit line of slope $\tau' = 1.50$ is shown. The expected value is $\tau' = 1.53$.

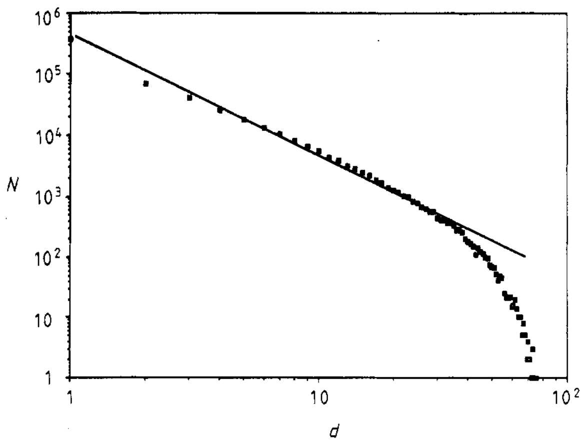


Figure 7. Log–log plot of the distance distribution between two consecutive growth sites. The distribution is cut off for large distances by the lattice size L, here 80. A straight line of slope 2 is shown for comparison with the theoretical expectation (equation (22)).

8. The case of model III

In order to see the implication of our discussion on the scaling behaviour of model III, we face the difficulty of first computing the fractal dimension D, which is apparently different from that of usual percolation (see above). We do not have at present any argument to get this exponent directly. However, it is well known that two exponents are sufficient to compute all others in usual percolation. Using the fact that the structure of the external perimeter should not be affected by the incompressibility effect (illustrated in figure 1(a) and 1(b), we should be able to compute all exponents we need using D as a given parameter and all critical exponents from percolation.

In particular, through the mapping we can relate the cluster size in the invasion percolation problem s_1 with the size of the corresponding cluster in percolation, s, by:

$$s_1{}^{1/D} = s^{1/D_f} \tag{28}$$

we thus introduce the 'translation' exponent, $t = D_f/D$†. Using $D = 1.82$, we get $t = 1.04$. Therefore, we get the distribution of clusters of size s_1, by the change of variable in (12) justified by the fact that the differential element $n_s\,\mathrm{d}s$ should correspond in the two cases:

$$n_{s_1} = s_1{}^{-t(\tau-1)-1} f(s_1^{t\sigma}(p - p_c)). \tag{29}$$

Upon performing the substitution $\tau \to t(\tau - 1) + 1$ and $\sigma \to t\sigma$, we obtain for τ' (and therefore τ'')

$$\tau'' \to t\{\tau'' - 1\} + 1 \tag{30}$$

thus, the exponent b appearing in (1) should again amount to

$$b = 1/D \tag{31}$$

or numerically, $b = 0.55$. This estimate is a little lower than, although consistent with, the $b = 0.6$ estimate of Furuberg *et al* [10]. We note that the scaling of $N(r, T)$ for the short-time behaviour, is similar to that obtained previously using (31).

Obviously, for the case of the exponent α, and thus a, no change from (23) is expected, provided the symbol D refers to the effective fractal dimension.

9. Conclusion

In conclusion, we have shown that the exponents appearing in the temporal development of invasion percolation can be expressed as a function of the critical exponents of usual percolation, for models I and II. For model III, where presumably the fractal dimension of the invaded cluster differs from that of the pure percolation case, this exponent can be readily included in our description to give the scaling form observed by Furuberg *et al* [10]. Numerical results are in good agreement with the theoretical scalings proposed.

We suggest that an experimental study of the above-mentioned properties could constitute an efficient tool to extract the pore-size distribution in a porous medium and, more importantly, to study correlations in size between neighbouring pores. This can be achieved by computing the burst-size distribution extracted from the record of pressure as a function of the injected volume.

† A similar result has been obtained independently by Furuberg (private communication).

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